$\mathcal{E}_2$-FORMALITY VIA OBSTRUCTION THEORY

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Abstract. We attack the question of $\mathcal{E}_2$-formality of differential graded algebras over $\mathbb{F}_p$ via obstruction theory. We are able to prove that $\mathcal{E}_2$-algebras whose cohomology ring is a polynomial algebra on even degree classes are intrinsically formal. As a consequence we prove $\mathcal{E}_2$-formality of the classifying space of some compact Lie group or of Davis-Januszkiewicz spaces.

Formality of spaces is an old idea originating in the field of rational homotopy theory. In that context, a space is said to be formal if its rational cohomology is quasi-isomorphic to its cochains as a commutative or $\mathcal{E}_\infty$-algebra. When this is the case, the whole rational homotopy type is controlled by a very manageable algebraic gadget.

With integral or torsion coefficients, the question of formality admits several versions. The most naive generalization (i.e. asking for cochains to be quasi-isomorphic to cohomology as $\mathcal{E}_\infty$-algebras) does not have interesting examples, and one is lead to studying weaker forms of formality. There is some literature devoted to proving $\mathcal{E}_1$-formality of certain spaces, i.e. proving that $C^*(X,k)$ is quasi-isomorphic to $H^*(X,k)$ as differential graded algebras (see for example [EH92, BB20, Sal20, DCH21, CH22]). When this is the case some invariants of X can be computed from the cohomology ring of X. For example, the bar construction spectral sequence

$$\mathrm{Tor}^{H^*(X;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) \implies H^*(\Omega X,\mathbb{F}_p)$$

collapses at the E_2-page. However, this collapse results is additive and there are usually some multiplicative extension that cannot be resolved by $\mathcal{E}_1$-formality.

There is in fact a countable family of formality properties interpolating between $\mathcal{E}_1$-formality and $\mathcal{E}_\infty$-formality. Namely, one can study $\mathcal{E}_n$-formality for any n. We say that a space X is $\mathcal{E}_n$-formal if its cohomology ring viewed as an $\mathcal{E}_n$-algebra using the map of operads $\mathcal{E}_n \to \mathcal{C}om$ is quasi-isomorphic to its singular cochains with its underlying $\mathcal{E}_n$-algebra structure. This notion was introduced by Mandell in [Man09]. He made several conjectures about it. In particular, he conjectured that n-fold suspensions are $\mathcal{E}_n$-formal. This conjecture was proved very recently in [HL24].

In the present paper, we study $\mathcal{E}_2$-formality. Our approach is obstruction theoretic. Obstruction theory has been classically used to prove or study formality (see for example [HS79, BB20, Sal17, Emp24]). Typically, there is a sequence of obstruction classes in Hochschild or André-Quillen cohomology groups that have to vanish for the algebra to be formal. In general, computing the actual obstructions can be very difficult unless the group in which it lives is zero.

In our case, we exploit the fact that the operations on the homology of an $\mathcal{E}_2$-algebras are very explicit thanks to the work of Cohen (see [CLM76]) and quite manageable. The obstruction group is a Quillen cohomology group in the category of W_1-algebras, where the monad W_1 is the monad of homology operations on $\mathcal{E}_2$-algebras. This makes the

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obstruction groups computable in certain easy situations. Our main result is Theorem 3.4 which states that $\mathcal{E}_2$-algebras whose cohomology ring is polynomial on even degree variables are $\mathcal{E}_2$-intrinsically formal.

We are in fact able to push this result a bit further to prove formality of certain diagrams of $\mathcal{E}_2$-algebras. As a corollary, we prove $\mathcal{E}_2$-formality of the classifying space of compact Lie groups at primes that do not divide the order of the Weyl group (Theorem 5.1) generalizing a recent result of Benson and Greenlees proving $\mathcal{E}_1$-formality of such spaces (see [BG23]). We recover $\mathcal{E}_2$-formality of Davis Januszkiewicz spaces (originally proven by Franz in [Fra21b]) and we prove a multiplicative collapse result for certain Eilenberg-Moore spectral sequences (Theorem 5.8).

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Conventions. We denote by $\mathbf{grVect}_k$ the category of graded vector spaces over a field k. This category has a symmetric monoidal structure whose symmetry isomorphism involves the usual sign. Our graded vector spaces are cohomologically graded. We write $V \mapsto sV$ for the shift in this category given by $(sV)^i = V^{i-1}$.

Given a graded vector space V, we denote by $\mathrm{Sym}(V)$ the symmetric algebra on V and $\Lambda(V)$ the exterior algebra on V. Of course, the sign rules implies that, if V is concentrated in odd degrees and the characteristic of the field is not 2, then $\mathrm{Sym}(V) = \Lambda(V)$.

1. Quillen cohomology

1.1. **Non-abelian derived functor.** We follow the treatment of [Fra15]. Let $\mathbf{C}$ be a complete and cocomplete category with a set of projective compact generators. Following Frankland, such a category shall be called quasi-algebraic. It is a theorem of Quillen that, if $\mathbf{C}$ is a quasi-algebraic category, its category of simplicial objects has a model structure transferred along the right adjoint functor

$$\begin{aligned} s\mathbf{C} &\to s\mathbf{Set}^{\mathcal{G}} \\ X_\bullet &\mapsto \{\mathrm{Hom}_{\mathbf{C}}(G, X_\bullet)\}_{G\in\mathcal{G}} \end{aligned}$$

where $\mathcal{G}$ is a set of compact projective generators of $\mathbf{C}$.

Given a left adjoint functor between two quasi-algebraic categories, we define its left derived functor

$$\mathrm{L}F : s\mathbf{C} \to s\mathbf{D}$$

to be the left derived functor in the sense of model categories of the functor $F : s\mathbf{C} \to s\mathbf{D}$ given by degreewise application of the functor F.

Remark 1.1. There is a conceptual description of the ∞-category underlying the model category $s\mathbf{C}$ as the *non-abelian derived category of* $\mathbf{C}$. Explicitly, this is completion of the category $\mathbf{C}^{cp}$ of compact projective objects of $\mathbf{C}$ under ∞-sifted colimits. Using this terminology, the functor $\mathrm{L}F$ is the unique extension of $F_{|\mathbf{C}^{cp}}$ into a functor that preserves sifted colimits. This is originally due to Lurie but we refer the reader to [ČS24, Section 5.1.1] for a very comprehensive description of this point of view.

1.2. **Quillen cohomology.** Let $\mathbf{C}$ is a quasi-algebraic category. For c an object of $\mathbf{C}$, we may consider the category $\mathbf{Ab}(\mathbf{C}/c)$ of abelian group objects over c. By the adjoint functor theorem, there is an abelianization functor

$$\mathrm{Ab} : \mathbf{C}/c \to \mathbf{Ab}(\mathbf{C}/c)$$

which is left adjoint to the forgetful functor. We define the cotangent complex of c to be the left derived functor of the abelianization functor. We denote this object by $L_c^{\mathbf{C}}$ or simply L_c if there is no ambiguity. It follows from [Fra15, Propositions 3.33 and 3.34] that the categories $\mathbf{C}/c$ and $\mathrm{Ab}(\mathbf{C}/c)$ are quasi-algebraic and hence that the relevant model structures exist.

Given an object $m \in \mathbf{Ab}(\mathbf{C}/c)$, we define the Quillen cohomology of c with coefficients in m as :

$$\mathrm{AQ}^s_{\mathbf{C}}(c, m) := \pi^s \mathrm{Hom}_{\mathbf{Ab}(\mathbf{C}/c)}(L_c, m).$$

(Observe that $\mathrm{Hom}_{\mathbf{Ab}(\mathbf{C}/c)}(L_c, m)$ is a cosimplicial abelian group, and we denote by π^s is the s-th cohomology group of its associated cochain complex.)

Remark 1.2. In all the cases of interest to us, the category $\mathbf{C}$ will come with a forgetful functor to the category of graded vector spaces over k. Moreover, it will be the case that the shift functor on the category of graded vector space will pass to the category of abelian group objects in $\mathbf{C}/c$. In this case, the André-Quillen cohomology groups are bigraded

$$\mathrm{AQ}^{s,t}_{\mathbf{C}}(c, m) = \mathrm{AQ}^s_{\mathbf{C}}(c, s^t M).$$

Proposition 1.3. *Let $F : \mathbf{C} \leftrightarrows \mathbf{D} : U$ be an adjunction between quasi-algebraic categories. Let c be an object of $\mathbf{C}$. Assume that $\mathrm{L}F(c) \to F(c)$ is a weak equivalence. Let $m \in \mathbf{Ab}(\mathbf{D}/F(c))$. Then there is an isomorphism*

$$\mathrm{AQ}^*_{\mathbf{D}}(Fc, m) \cong \mathrm{AQ}^*_{\mathbf{C}}(c, Um).$$

Proof. This is almost [Fra15, Proposition 4.10 (4)] except that we do not ask that F preserves all weak equivalences contrary to Frankland. However, it is straightforward to check that all that is needed in the proof is that $\mathrm{L}F(c) \to F(c)$ is a weak equivalence. ☐

2. Cohomology of $\mathcal{E}_2$-algebras

In this section, we restrict to working over a prime field $\mathbb{F}_p$ with p a prime number. We recall the work of Cohen (see [CLM76, Chapter III]) describing the cohomology operations on $\mathcal{E}_2$-algebras.

Definition 2.1. A shifted restricted Lie algebra is a graded vector space V^* with a Lie bracket

$$[-,-] : V^i \otimes V^j \to V^{i+j-1}$$

and a restriction

$$\xi : V^i \to V^{pi-p+1}$$

satisfying the following relations

(1) The Lie bracket is bilinear.
(2) The Lie bracket is antisymmetric

$$[x, y] = -(-1)^{(|x|-1)(|y|-1)}[y, x].$$

(3) The Lie bracket satisfies the graded Jacobi relation.

$$(-1)^{(|x|-1)(|z|-1)}[x,[y,z]] + (-1)^{(|x|-1)(|y|-1)}[y,[z,x]] + (-1)^{(|y|-1)(|z|-1)}[z,[x,y]] = 0$$

(4) The Lie bracket satisfies the relation $[x,[x,x]]=0$. (This relation follows from the Jacobi relation if $p\neq 3$.)
(5) The operation ξ is zero on even degree elements. (If $p=2$, this relation does not exist.)
(6) We have
$$\xi(x+y)=\xi(x)+\xi(y)+\sum_{i=1}^{p-1}d_2^i(x)(y)$$
where the operations d_2^i are described in [CLM76, p.218].
(7) We have
$$\xi(\lambda x)=\lambda\xi(x)$$
for all $\lambda\in\mathbb{F}_p$.
(8) We have
$$[x,\xi(y)]=\mathrm{ad}^p(y)(x)$$

Remark 2.2. Over $\mathbb{F}_2$, relation (6) becomes
$$\xi(x+y)=\xi(x)+\xi(y)+[x,y].$$
One consequence of this is that $[x,x]=0$ whatever the degree of x is. In other characteristics this only holds for elements of odd degree.

Equivalently a shifted restricted Lie algebra is a graded vector space V together with the data of a graded restricted Lie algebra structure on $s^{-1}V$. We denote by rLie**Alg** the category of restricted Lie algebras and rLie_1**Alg** the category of shifted restricted Lie algebras.

Definition 2.3. A W_1-algebra is a graded vector space over $\mathbb{F}_p$ equipped with

- A degree -1 Lie bracket
$$[-,-]:V^i\otimes V^j\to V^{i+j-1}$$
- A restriction
$$\xi:V^i\to V^{pi-p+1}$$
- An additional linear map
$$\zeta:V^i\to V^{pi-p+2}$$
(if $p=2$, this map does not exist)
- A product
$$V^i\otimes V^j\to V^{i+j}$$

such that the following axioms are satisfied.

(1) The Lie bracket and ξ make V into a shifted restricted Lie algebra.
(2) The product is bilinear and graded commutative.
(3) The operation ζ vanishes on even degree elements.
(4) We have the formula
$$[x,\zeta y]=0$$
(5) The bracket is a derivation with respect to the product in each variable.
$$[x,yz]=[x,y]z+(-1)^{|y|(|x|+1)}[x,z]y$$

(6) The operations ξ and ζ satisfy a Cartan formula.

$$\xi(xy) = x^p\xi(y) + \xi(x)y^p + \sum \Gamma_{i,j}x^iy^j$$
$$\zeta(xy) = \zeta(x)y^p - x^p\zeta(y)$$

where the term $\Gamma_{i,j}$ are described on page 335 of [CLM76]. (if $p = 2$ the Cartan formula is

$$\xi(xy) = x^2\xi(y) + \xi(x)y^2 + x[x, y]y)$$

Let us denote by

$$F^{\mathrm{W}_1} : \mathbf{grVect}_{\mathbb{F}_p} \to \mathrm{W}_1\mathbf{Alg}$$

the free W_1-algebra monad.

Theorem 2.4 (Cohen)**.** *Let C be a cochain complex of $\mathbb{F}_p$-vector spaces. Let $\mathcal{E}_2$ denotes the free $C_*(\mathcal{E}_2, \mathbb{F}_p)$-algebra monad. There is a natural isomorphism*

$$H^*(\mathcal{E}_2(C)) \cong F^{\mathrm{W}_1}(H^*(C))$$

In particular, the cohomology of a dg-$\mathcal{E}_2$-algebra is naturally a W_1-algebra.

By standard abstract nonsense, the restriction functor

$$\mathrm{W}_1\mathbf{Alg} \to \mathrm{rLie}_1\mathbf{Alg}$$

has a left adjoint that we shall denote by $F^{\mathrm{W}_1}_{\mathrm{rLie}_1}$.

We shall need an explicit description of this left adjoint. Given a graded vector space, V, we denote by ζV the graded vector space

$$\zeta V = \bigoplus_{i \text{ odd}} s^{i-pi+p-2}V^i$$

There is an operation

$$\zeta : V \to \zeta V$$

taking $v \in V^i$ with i odd to the element v in the summand $s^{i-pi+p-2}V^i$ of ζV. This operation is not a map of graded vector space, instead it behaves with respect to the degree as the operation ζ in a W_1-algebra. We define Sym_ζ to be the following functor from the category of graded vector spaces to itself

$$V \mapsto \mathrm{Sym}(V \oplus \zeta V).$$

Proposition 2.5. *Let p be an odd prime. The composed functor*

$$\mathrm{rLie}_1\mathbf{Alg} \xrightarrow{F^{\mathrm{W}_1}_{\mathrm{rLie}_1}} \mathrm{W}_1\mathbf{Alg} \xrightarrow{\text{forget}} \mathbf{grVect}_{\mathbb{F}_p}$$

is isomorphic to $\mathfrak{g} \mapsto \mathrm{Sym}_\zeta(\mathfrak{g})$. A similar proposition holds over $\mathbb{F}_2$ if we replace Sym_ζ by Sym.

Proof. Both functors of $\mathfrak{g}$ preserve ordinary sifted colimits. Moreover, $\mathrm{rLie}_1\mathbf{Alg}$ is an algebraic category, i.e., it is the completion of its subcategory of free algebras on a finite dimensional vector space under ordinary sifted colimits. Therefore, it suffices to prove that both functors coincide on the category of free shifted restricted Lie algebras on a finite dimensional graded vector space.

If $\mathfrak{g}$ is the free restricted Lie algebra on V, then a basis of $\mathfrak{g}$ is given by symbols $\xi^i l_\alpha$ where the symbols l_α form an arbitrary basis of the free Lie algebra on V and the exponent i is arbitrary if l_α is of even degree and is zero otherwise. On the other hand, $F^{\mathrm{W}_1}_{\mathrm{rLie}_1}(\mathfrak{g}) \cong F^{\mathrm{W}_1}(V)$ is explicitly described in [CLM76, page 227]. It is the free commutative algebra on elements of the form $\zeta^\epsilon \xi^i l_\alpha$ where l_α and ξ^i are as before and $\epsilon \in \{0,1\}$ and is required to be 0 if $\xi^i l_\alpha$ is of even degree. The result follows from this explicit description.

The case $p = 2$ is similar. □

Remark 2.6. Observe that if $\mathfrak{g}$ is concentrated in even degrees (in which case the Lie bracket and ξ must be zero for degree reasons), then $F^{\mathrm{W}_1}_{\mathrm{rLie}_1}(\mathfrak{g})$ is simply $\mathrm{Sym}(\mathfrak{g})$ with trivial operations $[-,-]$, ξ and ζ.

Proposition 2.7. *Let V be a graded vector space, then*

$$\mathrm{LSym}_\zeta(V) \simeq \mathrm{Sym}_\zeta(V)$$

and

$$\mathrm{LSym}(V) \simeq \mathrm{Sym}(V)$$

Proof. This proposition holds for any functor $F : \mathbf{grVect}_k \to \mathbf{grVect}_k$ that preserves filtered colimits. Indeed, $\mathrm{L}F$ coincides with F on finite dimensional vector spaces by construction, moreover, both $\mathrm{L}F$ and F preserve filtered colimits, it follows that they must coincide on any graded vector space. □

3. Computation of the obstruction group

Definition 3.1. We call a bigraded abelian group even (resp. odd) if it is concentrated in bidegrees (s,t) such that $s+t$ is even (resp. odd).

Lemma 3.2. *Let $A = \Lambda(V)$ be the exterior algebra on a graded vector space V concentrated in odd degrees and of finite total dimension. Let M be an A-module concentrated in odd degrees and degreewise of finite dimension. We view M as a bimodule using the multiplication map $A \otimes A \to A$. Then, the bigraded abelian group*

$$\mathrm{AQ}^{*,*}_{\mathrm{Ass}\mathbf{Alg}}(A,M)$$

is even.

Proof. There is an isomorphism

$$\begin{aligned}\mathrm{AQ}^{s,*}(A,M) &\cong \mathrm{Der}(A,M) \text{ if } s=0\\ &\cong \mathrm{HH}^{s+1}(A,M) \text{ if } s>0\end{aligned}$$

Since $\mathrm{Der}(A,M) \subset \mathrm{Hom}_k(A,M)$ is concentrated in even degrees, it suffices to prove that $\mathrm{HH}^{*,*}(A,M)$ is odd.

There exists a unique map of commutative algebras

$$A \to A \otimes A$$

sending $v \in V$ to $v \otimes 1 - 1 \otimes v$. Indeed, if k is of characteristic different from 2, then $\Lambda(V) \cong \mathrm{Sym}(V)$ is free and if k is of characteristic 2, then we do indeed have

$$(v \otimes 1 - 1 \otimes v)^2 = 0$$

By [CE56, Theorem X.6.1], we have an isomorphism

$$\mathrm{HH}^*(A,M) \cong \mathrm{Ext}^*_A(k,\tilde{M})$$

where M is viewed as a left A-module using the map above. By our assumption, $\tilde{M}$ is simply the k-vector space M with the trivial A-module structure. So, using the finite dimensionality of M, we have

$$\mathrm{Ext}^*_A(k,\tilde{M}) \cong \mathrm{Ext}^*_A(k,k) \otimes_k M$$

Since M is odd, it suffices to prove that $\mathrm{Ext}^*_A(k,k)$ is even. If V is one-dimensional so that $A = k[x]/x^2$. Then a free resolution of k as an A-module is given by

$$k \leftarrow A \xleftarrow{\times x} s^{|x|}A \xleftarrow{\times x} s^{2|x|}A \xleftarrow{\times x} \dots$$

It follows that the bigraded abelian group $\mathrm{Ext}^{*,*}_A(k,k)$ is even. For a general V, $\mathrm{Ext}_A(k,k)$ is a tensor product of finitely many bigraded abelian groups of this form, therefore the answer is still even. □

Lemma 3.3. *Let V be a graded vector space concentrated in even degrees viewed as a shifted restricted Lie algebra with trivial bracket and restriction, then $\mathrm{AQ}^{*,*}_{\mathrm{W}_1}(F^{\mathrm{W}_1}_{\mathrm{rLie}_1}(V), M)$ is even.*

Proof. By Proposition 1.3, Proposition 2.5 and Proposition 2.7, we have an isomorphism

$$\mathrm{AQ}^*_{\mathrm{W}_1}(F^{\mathrm{W}_1}_{\mathrm{rLie}_1}(V), M) \cong \mathrm{AQ}^*_{\mathrm{rLie}_1}(V, M) \cong \mathrm{AQ}^*_{\mathrm{rLie}}(s^{-1}V, s^{-1}M)$$

The universal enveloping algebra of the restricted Lie algebra $s^{-1}V$ is simply the exterior algebra $\Lambda(s^{-1}V)$ (indeed since $s^{-1}V$ is in odd degrees, the exterior algebra is simply the symmetric algebra when the characteristic is different from 2). Then we can use again Proposition 1.3 (which in this context is [DFI24, Theorem 14.2]), we thus have

$$\mathrm{AQ}^*_{\mathrm{rLie}}(s^{-1}V, s^{-1}M) \simeq \mathrm{AQ}^*_{\mathrm{Ass}}(\Lambda(s^{-1}V), s^{-1}M)$$

which is even by the previous lemma. □

Theorem 3.4. *Let A be a dg-$\mathcal{E}_2$-algebra over $\mathbb{F}_p$ such that $H^*(A) = \mathrm{Sym}(V)$ with V a finite dimensional graded vector space concentrated in even degrees. Let B be a dg-$\mathcal{E}_2$-algebra also concentrated in even degrees and degreewise finite dimensional. Then*

(1) for any map of $\mathbb{F}_p$-algebras

$$f : H^*(A) \to H^*(B)$$

there is a map in the homotopy category of $\mathcal{E}_2$-algebras

$$\tilde{f} : A \to B$$

such that $H^*(\tilde{f}) = f$.

(2) Any $\mathcal{E}_2$-algebra whose cohomology ring is isomorphic to the cohomology ring of A must be quasi-isomorphic to A.

Proof. First observe that (2) follows easily from (1). Thanks to Remark 2.6, a map of $\mathbb{F}_p$-algebras

$$f : H^*(A) \to H^*(B)$$

is automatically a map of W_1-algebras. We may thus use the spectral sequence of [JN14, Theorem 4.5] computing the mapping space $\mathrm{map}_{\mathcal{E}_2-\mathbf{Alg}}(A, B)$. The relevant obstructions to lifting f live in $E_2^{t,t-1}$. Thanks to [JN14, Theorem B], this group can be identified with $\mathrm{AQ}^t_{\mathrm{W}_1}(A, s^{t-1}H^*(B))$. The result then follows from the previous lemma. □

In the next section, we will push this result to certain diagrams of $\mathcal{E}_2$-algebras but we can already give one application originally due to Bayındır and Moulinos (see [BM22, Theorem 1.3]).

Theorem 3.5 (Bayındır, Moulinos)**.** *There is a weak equivalence of $\mathcal{E}_2$-algebras over $H\mathbb{F}_p$:*

$$\mathrm{THH}(\mathbb{F}_p) \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3$$

Proof. Since $S^3 \cong SU(2)$ is a Lie group, the space ΩS^3 is a 2-fold loop space. Therefore, both sides of this equation are $\mathcal{E}_2$-algebras in $H\mathbb{F}_p$-modules so they can be viewed as dg-$\mathcal{E}_2$-algebras. They also have isomorphic homotopy rings given by a polynomial ring on one generator of (homological) degree 2. □

Remark 3.6. In fact Bayındır and Moulinos prove that for any polynomial ring R over $\mathbb{F}_p$ on one even degree class, there is a unique $\mathcal{E}_2$-algebra with R as its homotopy ring (see [BM22, Theorem 2.1]). This result also follows from Theorem 3.4.

For the sake of completeness, we state and sketch the proof of the characteristic zero analogue of the above theorem.

Theorem 3.7. *Let A be a dg-$\mathcal{E}_2$-algebra over $\mathbb{Q}$ such that $H^*(A) = \mathrm{Sym}(V)$ with V a finite dimensional graded vector space concentrated in even degrees. Let B be a dg-$\mathcal{E}_2$-algebra also concentrated in even degrees and degreewise finite dimensional. Then*

(1) for any map of $\mathbb{Q}$-algebras
$$f : H^*(A) \to H^*(B)$$
there is a map in the homotopy category of $\mathcal{E}_2$-algebras
$$\tilde{f} : A \to B$$
such that $H^*(\tilde{f}) = f$.

(2) Any $\mathcal{E}_2$-algebra whose cohomology ring is isomorphic to the cohomology ring of A must be quasi-isomorphic to A.

Proof. This is completely analogous to the proof of Theorem 3.4 except that the monad W_1 has to be replaced by the Gerstenhaber monad. We obtain exactly as above an isomorphism
$$\mathrm{AQ}_{\mathrm{Ger}}(H^*(A), H^*(B)) \cong \mathrm{AQ}_{\mathrm{Ass}}(\Lambda(s^{-1}V), s^{-1}H^*(B)).$$
Moreover, the right hand side is even thanks to Lemma 3.2 (which does not depend on the characteristic of the field). □

Remark 3.8. The analogous theorem with $\mathcal{E}_2$ replaced by $\mathcal{E}_\infty$ is also true and well-known. Indeed, in that case, A can be strictified to a commutative algebra. Then we can produce a quasi-isomorphism
$$H^*(A) \to A$$
by sending each generator to a choice of cocycle representing it.

4. Diagrams of $\mathcal{E}_2$-algebras

4.1. **Main theorem.** Let $\mathbf{C}$ a quasi-algebraic category and let I be a small category. In this situation $\mathbf{C}^I$ is also a quasi-algebraic category. Given $c : I \to \mathbf{C}$ and $m \in \mathbf{Ab}(\mathbf{C}^I/c)$, we denote by $\mathrm{AQ}_{\mathbf{C},I}(c, m)$ the corresponding Quillen cohomology.

Proposition 4.1. *Let $V : I \to$ **grVect** be a diagram taking values in finite dimensional graded vector spaces concentrated in odd degrees. Let $M : I \to$ **grVect** be a $\Lambda(V)$-module satisfying pointwise the conditions of Lemma 3.2. We moreover assume that M viewed as a diagram of k-vector spaces is injective. Then $\mathrm{AQ}_{\mathrm{Ass},I}(\Lambda(V), M)$ is even.*

Proof. First recall that if $\mathbf{C}$ is an ∞-category and I a small category, we have
$$\mathrm{map}_{\mathbf{C}^I}(F, G) \simeq \mathrm{holim}_{\Delta}\left([s] \mapsto \prod_{i_0 \to i_1 \to \ldots \to i_s} \mathrm{map}(F(i_0), G(i_s))\right).$$

By definition for A an associative algebra in $\mathbf{grVect}^I$ and M an A-bimodule, we have

$$\mathrm{AQ}_{\mathrm{Ass},I}(A,M) = \pi_*\mathrm{map}^h_{s\mathrm{Ass}\mathbf{Alg}^I/A}(A \to A, A \oplus M \to A).$$

This mapping space is the homotopy fiber over the identity map of the map

$$\mathrm{map}^h_{s\mathrm{Ass}\mathbf{Alg}^I}(A, A \oplus M \to A) \to \mathrm{map}^h_{s\mathrm{Ass}\mathbf{Alg}^I}(A, A).$$

Combining this observation with the cosimplicial space descrpition above, we obtain a cosimplicial space whose totalization computes $\mathrm{AQ}_{\mathrm{Ass},I}(A,M)$ given by

$$[s] \mapsto \prod_{i_0 \to i_1 \to \ldots \to i_s} \mathrm{map}^h_{s\mathrm{Ass}\mathbf{Alg}/A(i_s)}(A(i_0) \to A(i_s), A(i_s) \oplus M(i_s) \to A(i_s)).$$

The E_1 page of the corresponding Bousfield-Kan spectral sequence is given by

$$E_1^{s,t} = \prod_{i_0 \to i_1 \to \ldots \to i_s} \mathrm{AQ}^t(A(i_0), M(i_s)).$$

From the comparision between Quillen cohomology and Hochschild cohomology, we see that the row $t = 0$ is given by the cosimplicial abelian group

$$[s] \mapsto \prod_{i_0 \to i_1 \to \ldots \to i_s} \mathrm{Der}(A(i_0), M(i_s)),$$

while for $t > 0$, we get

$$[s] \mapsto \prod_{i_0 \to i_1 \to \ldots \to i_s} \mathrm{HH}^{t+1}(A(i_0), M(i_s)).$$

So far we did not use anything about the specific situation and the above discussion would apply to any pair (A, M). Using the computation of Lemma 3.2, we see that the row $t < 0$ of the E_1-page is of the form

$$[s] \mapsto \prod_{i_0 \to i_1 \to \ldots \to i_s} \mathrm{Hom}_k(F(i_0), M(i_s)),$$

where $F : I \to \mathbf{grVect}_k$ is the dual of the functor

$$i \mapsto \mathrm{Ext}_{A(i)}(k, k).$$

Likewise the 0th row is simply given by the cosimplicial abelian group

$$[s] \mapsto \prod_{i_0 \to i_1 \to \ldots \to i_s} \mathrm{Hom}_k(V(i_0), M(i_s)).$$

(indeed there is an isomorphism $\mathrm{Der}(\Lambda(V), M) \cong \mathrm{Hom}_k(V, M)$). From this observation, using injectivity of M, we deduce that the E_2-page of the spectral sequence is concentrated on the column $s = 0$ and

$$E_2^{0,-t} = \ker\left(d_1 : \prod_{i \in I} \mathrm{AQ}^t(A(i), M(i)) \to \prod_{f : i \to j} \mathrm{AQ}^t(A(i), M(j))\right).$$

In particular, we see that

$$\mathrm{AQ}_I^{*,*}(A, M) \subset \prod_{i \in I} \mathrm{AQ}^{*,*}(A(i), M(i))$$

and is therefore even by Lemma 3.2. □

Corollary 4.2. *Let $V : I \to \mathbf{grVect}_{\mathbb{F}_p}$ be a diagram of finite dimensional graded vector space concentrated in even degrees. Let $M : I \to \mathbf{grVect}_{\mathbb{F}_p}$ be an injective diagram concentrated in even degrees and equipped with the structure of a module over $F^{\mathrm{W}_1}_{\mathrm{rLie}_1}(V)$. Then $\mathrm{AQ}^{*,*}_{\mathrm{W}_1,I}(F^{\mathrm{W}_1}_{\mathrm{rLie}_1}(V), M)$ is even.*

Proof. As in Lemma 3.3, we can reduce to showing that $\mathrm{AQ}_{Ass,I}(\Lambda(s^{-1}V), s^{-1}M)$ is even which is the content of the previous proposition. $\square$

Theorem 4.3. *Assume that $i \mapsto A(i)$ is a diagram of differential graded $\mathcal{E}_2$-algebra over $\mathbb{F}_p$ such that*

(1) There is a diagram $V : I \to \mathbf{grVect}_{\mathbb{F}_p}$ which is objectwise finite dimensional and concentrated in even degrees, and a natural isomorphism
$$H^*(A(i)) \cong F^{\mathrm{W}_1}_{\mathrm{rLie}_1}(V(i))$$
(2) The diagram
$$i \mapsto H^*(A(i))$$
is injective as an I-diagram in $\mathbb{F}_p$-vector spaces.

Then the diagram A is formal as a diagram of $\mathcal{E}_2$-algebras.

Proof. As in Theorem 3.4, we use obstruction theory. We observe that the hypothesis of [JN14, Theorem B] apply. The category $\mathcal{D}$ is simply the category of I-diagrams of graded vector spaces. Then conditikon a [JN14, Theorem B] holds thanks to our injectivity assumption. The monad T_{alg} is simply F^{W_1} applied objectwise. It follows that the relevant obstruction group is $\mathrm{AQ}^{t,t-1}_{\mathrm{W}_1,I}(F^{\mathrm{W}_1}_{\mathrm{rLie}_1}(V), H^*(A))$ which is zero thanks to the previous corollary. $\square$

4.2. **Criterion for injectivity.** We borrow the following definition from Hovey (see [Hov99, Definition 5.1.1])

Definition 4.4. A direct category is a category I with a functor $\lambda : I \to (\mathbb{N}, \leqslant)$ such that $\lambda(f) = \mathrm{id}$ if and only if $f = \mathrm{id}$.

Proposition 4.5. *Let I be a direct category. Consider a diagram $F : I^{\mathrm{op}} \to \mathbf{grVect}$. Then F is injective if for all $i \in I$ the canonical map*
$$F(i) \to \lim_{j \in I, \lambda(j) < \lambda(i)} F(j)$$
is surjective.

Proof. This is very similar to [Hov99, Proposition 5.1.4]. Let us call the number $\lambda(i)$ the "degree" of the object i. Given an objectise injective map $f : M \to N$ in $\mathbf{grVect}^I$, we can lift a map $g : M \to F$ to a map $\tilde{g} : N \to F$ inductively on degree. Assuming the lift has been produced up to degree n, the next step is to find a lift on an object i of degree $n+1$. This amounts to finding a diagonal filler in the following diagram

$$\begin{tikzcd} M(i) \arrow[r, "g(i)"] \arrow[d, "f(i)"'] & F(i) \arrow[d] \\ N(i) \arrow[r] & \lim_{\lambda(k)<\lambda(i)} F(k) \end{tikzcd}$$

in which the bottom horizontal map is the composite
$$N(i) \to \lim_{\lambda(k)<\lambda(i)} N(k) \xrightarrow{\tilde{g}} \lim_{\lambda(k)<\lambda(i)} F(k)$$

This lifting problem has a solution since the left vertical map is injective while the right vertical map is surjective. Once these liftings have been chosen for each i of degree $n+1$, they are automatically compatible since there are no non-identity maps between objects of the same degree. This completes the induction step. □

Example 4.6. Let $I = 0 \to 1$. We can consider I as a direct category with $\lambda(0) = 0$, $\lambda(1) = 1$. The proposition says that an arrow $f : M_1 \to M_0$ viewed as an I-diagram in **grVect** is injective if f is surjective. Similarly, a span diagram

$$M \to N \leftarrow P$$

is injective if each of the map is surjective.

Example 4.7. Let I be the poset of faces of a simplicial complex. Then $F : I^{\mathrm{op}} \to \mathbf{grVect}$ is injective if it is fat in the sense of [NR05, Definition 3.6]. Indeed, we can view I as a direct category with $\lambda : I \to \mathbb{N}$ the dimension function.

5. Applications

5.1. Formality of BG for some compact Lie groups.

Theorem 5.1. *Let G be a compact Lie group with maximal torus T and assume that p does not divide the order of $N_G(T)/T$. Then $C^*(BG, \mathbb{F}_p)$ is $\mathcal{E}_2$-formal.*

Proof. Let us write $W = N_G(T)/T$. In this situation, by [Fes81, Theorem 1.5] there is a quasi-isomorphism of $\mathcal{E}_2$-algebra

$$C*(BG; \mathbb{F}_p) \to C^*(BT, \mathbb{F}_p)^W \simeq C^*(BT, \mathbb{F}_p)^{hW}$$

similarly, there is an isomorphism of commutative algebras

$$H^*(BG; \mathbb{F}_p) \to H^*(BT, \mathbb{F}_p)^W$$

Since $H^*(BT; \mathbb{F}_p)$ is polynomial on even degree generators, then the result will hold if the fomality quasi-isomorphism

$$C^*(BT; \mathbb{F}_p) \simeq H^*(BT; \mathbb{F}_p)$$

given by Theorem 3.4 can be made W-equivariant. Since by assumption, p does not divide the order of W, it follows that any $\mathbb{F}_p[W]$-vector space is injective as a W-diagram and thus the result follows from Theorem 4.3. □

Remark 5.2. The fact that, under these assumptions $C^*(BG, \mathbb{F}_p)$ is formal as an $\mathcal{E}_1$-algebra is a theorem of Benson and Greenlees (see [BG23]).

Remark 5.3. It is observed by Benson and Greenlees in [BG23] that a compact Lie group satisfying the assumptions of the above theorem does not necessarily have polynomial cohomology. They give the example of the non-connected group $G = \mathbb{Z}/2 \ltimes T^2$ with $\mathbb{Z}/2$ acting on the torus by inversion. In that case $H^*(BG, \mathbb{F}_3)$ is not a polynomial algebra. It is given by the subalgebra $\mathbb{F}_3[x^2, xy, y^2]$ of $H^*(BT^2, \mathbb{F}_3) \cong \mathbb{F}_3[x, y]$. Neverthess BG is $\mathcal{E}_2$-formal over $\mathbb{F}_3$ thanks to our theorem.

Remark 5.4. The careful reader will notice that we did not exactly use the fact that p does not divide the order of $W = N_G(T)/T$. What matters for the proof is that the action of W make $H^*(BT, \mathbb{F}_p)$ into an injective $\mathbb{F}_p[W]$-module. For example, if $G = U(n)$, then $T = U(1)^n$ and $W = S_n$. But $H^*(BT, \mathbb{F}_p)$ is injective for all values of p. So we can deduce from our theorem that $BU(n)$ is $\mathcal{E}_2$-formal. Of course, this is not very interesting as $H^*(BU(n), \mathbb{F}_p)$

is a polynomial algebra on even degree generators so its formality also follows from Theorem 3.4. We do not know examples for which this extra generality is useful.

5.2. **Formality of Davis-Januszkiewicz spaces.** For our next application, recall, the given a relative CW-complex $A \subset X$ and a simplicial complex K with set of vertices V, we may form the polyhedral product $Z(K;(A,X)) \subset X^V$ given as

$$Z(K;(A,X)) = \bigcup_{\sigma\in K} X^\sigma \times A^{V-\sigma}.$$

A case of particular interest is the case $A = pt$ and $X = BS^1$. The resulting space is then called a Davis-Januszkiewicz space.

Theorem 5.5. *Let G be a compact Lie group such that the $\mathbb{F}_p$-cohomology of BG is polynomial on even degree generators. Then $C^*(Z(K;(pt,BG)),\mathbb{F}_p)$ is $\mathcal{E}_2$-formal for any simplicial complex K. Moreover we have an isomorphism of algebras*

$$H^*(Z(K;(pt,BG),\mathbb{F}_p) \cong \lim(\sigma \mapsto H^*(BG^\sigma,\mathbb{F}_p)).$$

Proof. This is an application of Theorem 4.3. In this case the diagram

$$\sigma \mapsto H^*(BG^\sigma,\mathbb{F}_p)$$

is injective by [NR05, Lemma 3.8] and Remark 4.7. It follows that the diagram $K^{\mathrm{op}} \to \mathbf{Alg}_{\mathcal{E}_2}$

$$\sigma \mapsto C^*(BG^\sigma,\mathbb{F}_p)$$

is formal, therefore, we have a quasi-ismorphism of $\mathcal{E}_2$-algebra

$$\operatorname{holim}(\sigma \mapsto H^*(BG^\sigma,\mathbb{F}_p)) \simeq \operatorname{holim}(\sigma \mapsto C^*(BG^\sigma,\mathbb{F}_p)) \simeq C^*(Z(K;(pt,BG));\mathbb{F}_p).$$

□

Remark 5.6. The case $G = S^1$ is a theorem of Matthias Franz (see [Fra21b]). Franz phrases his result using the notion of "homotopy Gerstenhaber algebras" insead of $\mathcal{E}_2$-algebras but we have learned from Anibal Medina-Mardones that the two results should be equivalent. Let us also mention that in that case, the cohomology of $Z(K;(pt,BS^1))$ can be computed and is given by the Stanley-Reisner algebra associated to the simplicial complex K.

Remark 5.7. The polyhedral product construction can be extended to any map $A \to X$ (not just relative CW-complexes) by replacing the definition above by the homotopy colimit of the diagram

$$\sigma \mapsto X^\sigma \times A^{V-\sigma}.$$

In particular, if G is a compact Lie group, we may form $Z(K;(G,pt))$. This spaces inherits a G^V-action and we have

$$Z(K;(G,pt))_{hG^V} \simeq (Z(K;(pt,BG)))$$

by [DS07, Lemma 2.3.2].

5.3. **Multiplicative collapse of some Eilenberg-Moore spectral sequences.**

Theorem 5.8. *Let $X \to B \leftarrow Y$ be a diagram of spaces. Assume that*

(1) The cohomology of all three spaces with $\mathbb{F}_p$ coefficients is a polynomial algebra on finitely many even degree generators.
(2) The maps $H^*(B,\mathbb{F}_p) \to H^*(X,\mathbb{F}_p)$ and $H^*(B,\mathbb{F}_p) \to H^*(Y,\mathbb{F}_p)$ are surjective and send generators to linear combinations of generators.

Then, under the classical convergence assumption, the Eilenberg-Moore spectral sequence collapses multiplicatively, i.e. there is an isomorphism of $\mathbb{F}_p$-algebras

$$\mathrm{Tor}^{H^*(B,\mathbb{F}_p)}(H^*(X,\mathbb{F}_p), H^*(Y,\mathbb{F}_p)) \cong H^*(X \times^h_B Y, \mathbb{F}_p).$$

Proof. The surjectivity assumption insures that the diagram

$$H^*(X,\mathbb{F}_p) \leftarrow H^*(B,\mathbb{F}_p) \rightarrow H^*(Y,\mathbb{F}_p)$$

is injective by Remark 4.6. From Theorem 4.3, we obtain that the diagram of $\mathcal{E}_2$-algebras

$$C^*(X,\mathbb{F}_p) \leftarrow C^*(B,\mathbb{F}_p) \rightarrow C^*(Y,\mathbb{F}_p)$$

is formal. It follows that there is a quasi-isomorphism of $\mathcal{E}_1$-algebras

$$H^*(X,\mathbb{F}_p) \otimes^{\mathrm{L}}_{H^*(B,\mathbb{F}_p)} H^*(Y,\mathbb{F}_p) \simeq C^*(X,\mathbb{F}_p) \otimes^{\mathrm{L}}_{C^*(B,\mathbb{F}_p)} C^*(Y,\mathbb{F}_p).$$

Under the Eilenberg-Moore convergence assumption, the right-hand side is quasi-isomorphic to $C^*(X \times^h_B Y, \mathbb{F}_p)$. □

Remark 5.9. Arguably the most interesting case of application of the previous theorem is when X, B and Y are classifying space of compact Lie groups. In that case this Theorem is a weaker version of the main results of [Fra21a, Car23]. Indeed the main theorem in those papers does not have our second assumption. On the other hand, those papers assume that $p \neq 2$ and we are able to say something also in the case $p = 2$ (see next remark).

Remark 5.10. Consider the diagonal inclusion $U(1) \to U(n)$. The quotient is $PU(n)$. By [Bau68, Section 8, Example 4], the induced map in cohomology

$$H^*(U(n),\mathbb{F}_2) \to H^*(U(1),\mathbb{F}_2)$$

is surjective if and only if n is odd. So in those cases, our theorem states that the Eilenberg-Moore spectral sequence computing $H^*(PU(n),\mathbb{F}_2)$ collapses multiplicatively. In contrast, if $n \equiv 2 \bmod 4$ this spectral sequence is known to have multiplicative extensions. In particular, if $n = 2$, there is a homeomorphism $PU(2) \cong \mathbb{RP}^3$ and it is observed in [Fra21a, Remark 12.9] that the algebra $\mathrm{Tor}^{H^*(BU(2),\mathbb{F}_2)}(H^*(BU(1),\mathbb{F}_2),\mathbb{F}_2)$ contains a non-zero element in degree 1 that squares to zero.

Corollary 5.11. *Let (X, x) be a based space with $H^*(X,\mathbb{F}_p) \cong \mathrm{Sym}(V)$ with V finite dimensional an concentrated in even positive degrees. Then there is an isomorphism of Hopf algebras*

$$H^*(\Omega X,\mathbb{F}_p) \cong \Lambda(s^{-1}V)$$

Proof. The previous theorem gives us an isomorphism of algebras

$$H^*(\Omega X,\mathbb{F}_p) \cong \mathrm{Tor}^{\mathrm{Sym}(V)}(\mathbb{F}_p,\mathbb{F}_p) = \Lambda(s^{-1}V)$$

On the other hand, by the previous theorem, the diagram

$$C^*(pt,\mathbb{F}_p) \leftarrow C^*(X,\mathbb{F}_p) \rightarrow C^*(pt,\mathbb{F}_p)$$

is $\mathcal{E}_2$ (and hence $\mathcal{E}_1$) formal. Equivalently, the augmented algebra $H^*(X,\mathbb{F}_p)$ is $\mathcal{E}_1$-formal. This implies that there is an isomorphism of coalgebras

$$H^*(\Omega X,\mathbb{F}_p) \cong H^*(\mathrm{Bar}(H^*(X,\mathbb{F}_p))$$

where Bar denotes the bar construction of an augmented algebra :

$$\mathrm{Bar}(A) := \mathbb{F}_p \otimes^{\mathrm{L}}_A \mathbb{F}_p.$$

□

Remark 5.12. As we mentioned in the proof of the corollary, the statement about the coalgebra structure only requires $\mathcal{E}_1$-formality. This holds if the cohomology is polynomial without the evenness assumption (see [SH70]). On the other hand, the statement about the algebra structure is not true if we only assume $\mathcal{E}_1$-formality. As an example of this phenomenon consider $X = K(\mathbb{Z}/2, 2)$. Then we have

$$H^*(X, \mathbb{F}_2) = \mathbb{F}_2[x_{2^n+1}, n \geqslant 0]$$

with x_2 denoting the fundamental class and

$$x_{2^n+1} = \mathrm{Sq}^{2^{n-1}} \ldots \mathrm{Sq}^1 x_2.$$

Then $C^*(X, \mathbb{F}_2)$ is $\mathcal{E}_1$-formal since its cohomology is polynomial.

On the other hand, $H^*(\Omega X, \mathbb{F}_2) = H^*(\mathbb{RP}^\infty, \mathbb{F}_2) = \mathbb{F}_2[y]$ with $|y| = 1$ with the Hopf algebra structure given by

$$\Delta(y) = y \otimes 1 + 1 \otimes y.$$

We observe that, as a coalgebras, there is indeed an isomorphism

$$H^*(\Omega X, \mathbb{F}_2) \cong \bigotimes_{n \geqslant 0} \Lambda[y_{2^n}] = H^*(\mathrm{Bar}(H^*(X, \mathbb{F}_2)))$$

but this isomorphism is not compatible with the algebra structure.

In fact, it can be shown that $C^*(X, \mathbb{F}_2)$ is not $\mathcal{E}_2$-formal. Indeed, for any space Y, the operation ξ of the W_1-structure on $H^*(Y, \mathbb{F}_2)$ is simply given by $x \mapsto \mathrm{Sq}_1(x) = \mathrm{Sq}^{|x|-1}(x)$. It follows that an $\mathcal{E}_2$-formal space must have a trivial operation Sq_1. This is not the case for X.

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